Chapter 4

Factoring in the First and Second Degrees

In This Chapter

- Dividing out a linear factor
- Tackling the factorization of quadratic expressions
- Watching for special cases in factoring

In this chapter, you discover how to change from several terms to one, compact product. The factoring patterns you see here carry over somewhat in more complicated expressions.

When factoring, you find linear expressions (such as $2xy + 4xz$) and quadratic expressions (such as $3x^2 - 12$ or $-16t^2 + 32t + 11$). These are *expressions* because they're made up of two or more terms with plus (+) or minus (–) signs between them.

Some quadratic expressions may have one variable in them, such as $2x^2 - 3x + 1$. Others may have two or more variables, such as $\pi r^2 + 2\pi rh$.

These expressions all have their place in mathematics and science. In this chapter, you see how they work for you and how to factor them.

Making Factoring Work

Factoring is another way of saying: "Rewrite this so everything is all multiplied together." You usually start out with an expression of two or more terms and have to determine how to rewrite them so they're all multiplied together in some way or another. And, oh yes, the two expressions have to be equal!

Facing the factoring method

Factoring is the opposite of distributing; it's "undistributing." When performing distribution, you multiply a series of terms by a common multiplier. Now, by factoring, you seek to find what a series of terms have in common and then move it, dividing the common factor or multiplier out from each term.

An expression can be written as the product of the largest number that divides all the terms evenly times the results of the divisions: $ab + ac + ad = a(b + c + d)$.

The absolutely *proper* way to factor an expression is to write the prime factorization of each of the numbers and look for the greatest common factor (GCF), which is the largest possible divisor shared by all the terms. What's really more practical and quicker in the end is to look for the biggest factor that *you can easily recognize.* Factor it out and then see if the numbers in the parentheses need to be factored again. Repeat the division until the terms in the parentheses are relatively prime.

Here's how to use the repeated division method to factor the expression $450x + 540y - 486z + 216$. You see that the coefficient of each term is even, so divide each term by 2:

$$450x + 540y - 486z + 216 =$$
$$2(225x + 270y - 243z + 108)$$

The numbers in the parentheses are a mixture of odd and even, so you can't divide by 2 again. But the numbers in the parentheses are all divisible by 9. So, factoring out 9:

$$2(225x + 270y - 243z + 108) =$$
$$2[9(25x + 30y - 27z + 12)]$$

Now multiply the 2 and 9 together to get

$$450x + 540y - 486z + 216 =$$
$$18(25x + 30y - 27z + 12)$$

You could've divided 18 into each term in the first place, but not many people know the multiplication table of 18. (It's a stretch even for me.) Looking at the terms in the parentheses, there's no single factor that divides *all* the coefficients evenly. The four coefficients are *relatively prime* (there is no one factor that they all share except the number 1), so you're finished with the factoring.

Factoring out numbers and variables

Variables represent values; variables with exponents represent the powers of those same values. For that reason, variables as well as numbers can be factored out of the terms in an expression, and in this section I tell you how.

When factoring out powers of a variable, the smallest power that appears in any one term is the most that can be factored out. For example, in an expression such as $a^4b + a^3c + a^2d + a^3e^4$, the smallest power of a that appears in any term is the second power, a^2. So you can factor out a^2 from all the terms because a^2 is the GCF. You can't factor anything else out of each term: $a^4b + a^3c + a^2d + a^3e^4 = a^2(a^2b + a^1c + d + a^1e^4)$.

Factor the GCF out of the expression $x^2y^3 + x^3y^2z^4 + x^4yz$.

$$x^2y^3 + x^3y^2z^4 + x^4yz = x^2y(y^2 + x^1y^1z^4 + x^2z)$$

The GCF is x^2y.

The real test of the factoring process is combining numbers and variables, finding the GCF, and factoring successfully.

Factor $12x^2y^3z + 18x^3y^2z^2 - 24xy^4z^3$.

Each term has a coefficient that's divisible by 2, 3, and 6. You select 6 as the largest of those common factors.

Each term has a factor of x. The powers on x are 2, 3, and 1. You have to select the *smallest* exponent when looking for the GCF. That's 1, so the common factor is just x.

Each term has a factor of y. The exponents are 3, 2, and 4. The smallest exponent is 2, so the common factor is y^2.

Each term has a factor of z, and the exponents are 1, 2, and 3. The number 1 is smallest, so you can pull out a z from each term.

Put all the factors together, and you get that the GCF is $6xy^2z$. So,

$$12x^2y^3z + 18x^3y^2z^2 - 24xy^4z^3 =$$
$$6xy^2z(2x^1y^1 + 3x^2z^1 - 4y^2z^2)$$

Doing a quick check, multiply through by the GCF in your head to be sure that the products match the original expression. Then do a sweep to be sure that there isn't a common factor among the terms within the parentheses.

Factor $-4ab - 8a^2b - 12ab^2$.

The coefficient of each term in the expression is negative; dividing out the negative from all the terms in the parentheses makes them positive.

$$-4ab - 8a^2b - 12ab^2 = -4ab(1 + 2a^1 + 3b^1)$$

When factoring out a negative factor, be sure to change the signs of each of the terms in the parentheses.

Getting at the Basic Quadratic Expression

The word *quadratic* refers to an expression that contains a power of 2. Yes, the prefix *quad* means four, but the origin of the word *quadratic* comes from the two dimensions of a square (which is four-sided).

The quadratic, or second-degree, expression in x has the x variable that is squared, and no x terms with powers higher than 2. The coefficient on the squared variable is not equal to 0. The standard quadratic form is $ax^2 + bx + c$.

By convention, the terms are usually written with the second-degree term first, the first-degree term next, and the number last. After you find the variable that's squared, write the rest of the expression in decreasing powers of that variable.

Rewrite $aby + cdy^2 + ef$ using the standard convention involving order. This is a second-degree expression in y.

Written in the standard form for quadratics, $ax^2 + bx + c$, where the second-degree term comes first, it looks like

$$cdy^2 + aby + ef$$

Following Up on FOIL and unFOIL

FOIL is an acronym used to help you multiply two binomials together by distributing. The *F* stands for *first; O,* for *outer; I,* for *inner;* and *L,* for *last.* This section involves undoing what is done by FOILing — you see how to factor a quadratic back into the two binomials.

Many quadratic expressions, such as $6x^2 + 7x - 3$, are the result of multiplying two *binomials* (expressions with two terms), so you can undo the multiplication by factoring them:

$$6x^2 + 7x - 3 = (2x + 3)(3x - 1)$$

The product of the binomials results in the trinomial, because, using FOIL, the product of the two first terms, the $2x$ and the $3x$, is $6x^2$. Then the product of the two outer terms, $2x$ and -1, is $-2x$. Add that to the product of the two inner terms, $9x$, and you get $7x$. The final term is the -3, which comes from the product of the two last terms.

When you look at an expression such as $2x^2 - 5x - 12$, you may think that figuring out how to factor this into the product of two binomials is an awful chore. And you may wonder whether it can even be factored that way. The nice thing in solving this particular puzzle is that there's a system to make unFOILing simple. You go through the system, and it helps you find what the answer is or even helps you determine if there isn't an answer. This can't be said about all factoring problems, but it is true of quadratics in the form $ax^2 + bx + c$. That's why quadratics are so nice to work with in algebra.

The key to unFOILing these factoring problems is being organized:

- Be sure you have an expression in the form $ax^2 + bx + c$.
- Be sure the terms are written in the order of decreasing powers.
- If needed, review a listing of prime numbers and perfect squares.
- Follow the steps.

Follow these steps to factor the quadratic $ax^2 + bx + c$, using unFOIL.

1. **Determine all the ways you can multiply two numbers to get *a*.**

   Every number can be written as at least one product, even if it's only the number times 1. So assume that there are two numbers, e and f, whose product is equal to a. These are the two numbers you want for this problem.

2. **Determine all the ways you can multiply two numbers together to get *c*.**

   If the value of c is negative, ignore the negative sign for the moment. Concentrate on what factors result in the absolute value of c.

   Now assume that there are two numbers, g and h, whose product is equal to c. Use these two numbers for this problem.

3. **Now look at the sign of *c* and your lists from steps 1 and 2.**

   - *If* c *is positive,* find a value from your Step 1 list and another from your Step 2 list such that the sum of their product and the product of the two numbers they're paired with in those steps results in b.

     Find $e \cdot g$ and $f \cdot h$, such that $e \cdot g + f \cdot h = b$.

- *If* c *is negative,* find a value from your Step 1 list and another from your Step 2 list such that the difference of their product and the product of the two numbers they're paired with from those steps results in *b*.

  Find $e \cdot g$ and $f \cdot h$, such that $e \cdot g - f \cdot h = b$.

4. **Arrange your choices as binomials.**

   The *e* and *f* have to be in the first positions in the binomials, and the *g* and *h* have to be in the last positions. They have to be arranged so the multiplications in Step 3 have the correct outer and inner products.

   $(e \quad h)\ (f \quad g)$

5. **Place the signs appropriately.**

   The signs are both positive if *c* is positive and *b* is positive.

   The signs are both negative if *c* is positive and *b* is negative.

   One sign is positive and one sign is negative if *c* is negative; deciding where to put the negative in the factorization depends on whether *b* is positive or negative and how you arranged the factors.

Factor the quadratic $2x^2 - 5x - 12$ using unFOIL.

1. **Determine all the ways you can multiply two numbers to get *a, which is 2 in this problem.***

   The number 2 is prime, so the only way to multiply and get 2 is $2 \cdot 1$.

2. **Determine all the ways you can multiply two numbers to get *c*, which is –12 in this problem.**

   Ignore the negative sign right now. The negative becomes important in the next step. Just concentrate on what multiplies together to give you 12.

   There are three ways to multiply two numbers together to get 12: $12 \cdot 1$, $6 \cdot 2$, and $4 \cdot 3$.

3. **Look at the sign of *c* and your lists from steps 1 and 2.**

   Because *c* is negative, you find a value from Step 1 and another from Step 2 such that the difference of their product and the product of the other numbers in the pairs results in b, which is –5 in this problem.

Use the 2 · 1 from Step 1 and the 4 · 3 from Step 2. Multiply the 1 from Step 1 times the 3 from Step 2 and then multiply the 2 from Step 1 times the 4 from Step 2.

$$(1)(3) = 3 \text{ and } (2)(4) = 8$$

The two products are 3 and 8, whose difference is 5.

4. **Arrange the choices in binomials.**

   The following arrangement multiplies the $(1x)(2x)$ to get the $2x^2$ needed for the first product. Likewise, the 4 and 3 multiply to give you 12. The outer product is $3x$ and the inner product is $8x$, giving you the difference of $5x$.

   $$(1x\quad 4)(2x\quad 3)$$

5. **Place the signs to give the desired results.**

   You want the $5x$ to be negative, so you need the $8x$ product to be negative. The following arrangement accomplishes this:

   $$(1x - 4)(2x + 3) =$$

   $$2x^2 + 3x - 8x - 12 =$$

   $$2x^2 - 5x - 12$$

In the next example, all the terms are positive. The sum of the outer and inner products will be used. And there are several choices for the multipliers.

Factor: $10x^2 + 31x + 15$.

1. **Determine all the ways you can multiply two numbers to get 10.**

   The 10 can be written as 10 · 1 or 5 · 2.

2. **Determine all the ways you can multiply two numbers to get 15.**

   The 15 can be written as 15 · 1 or 5 · 3.

3. **The last term is +15, so you want the sum of the products to be 31.**

   Using the $5 \cdot 2$ and the $5 \cdot 3$, multiply $2 \cdot 3$ to get 6, and multiply $5 \cdot 5$ to get 25. The sum of 6 and 25 is 31.

4. **Arrange your choices in the binomials so the factors line up the way you want to give you the products.**

   $(2x \quad 5)(5x \quad 3)$

5. **Placing the signs is easy, because everything is positive.**

   $(2x + 5)(5x + 3) =$

   $10x^2 + 6x + 25x + 15 =$

   $10x^2 + 31x + 15$

Making UnFOIL and the GCF Work Together

A quadratic, such as $40x^2 - 40x - 240$, can be factored using two different techniques, unFOILing and the GCF, which can be done in two different orders. One of the choices makes the problem easier. It's the order in which the factoring is done that makes one way easier and the other way harder. You just have to hope that you recognize the easier way before you get started.

If you should choose to use unFOIL, first, you have to deal with the four different ways of factoring 40 and the ten ways of factoring 240, finally giving you (after much computing) $(4x - 12)(10x + 20)$. And then you get to factor each of the two binomials, resulting in $40(x - 3)(x + 2)$. It took two types of factorization: unFOILing and taking out a GCF.

An easier way is to factor out the GCF *first.*

Factor $40x^2 - 40x - 240$ by using the GCF first.

Each term's coefficient is evenly divisible by 40. Doing the factorization:

$$40x^2 - 40x - 240 = 40(x^2 - x - 6)$$

Now, looking at the trinomial in the parentheses:

1. **Use unFOIL to factor the trinomial $x^2 - x - 6$.**

   Notice how the list of choices for factors of the coefficients is much shorter and more manageable than if you try to unFOIL before factoring out the GCF.

2. **Looking at the sign of the last term, –6, choose your products to create a difference of 1.**

   The middle term, x, is negative, so you want the $3x$, the product of the inner terms, to be negative. Finish the factoring. Then put the 40 that you factored out in the first place back into the answer.

   $$40x^2 - 40x - 240 = 40(x - 3)(x + 2)$$

Getting the Best of Binomials

If a *binomial* (two-term) expression can be factored at all, it will be factored in one of four ways. First, look at the addition or subtraction sign that always separates the two terms within a binomial. Then look at the two terms. Are they squares? Are they cubes?

Here are the four ways to factor a binomial:

- Finding the GCF
- Factoring the difference of two perfect squares
- Factoring the difference of two perfect cubes
- Factoring the sum of two perfect cubes

When you have a factoring problem with two terms, you can go through the list to see which method works. You see how to divide out a GCF earlier in this chapter, so here I show you the other three methods.

Facing up to the difference of two perfect squares

If two terms in a binomial are perfect squares and they're separated by subtraction, then the binomial can be factored. To factor one of these binomials, just find the square roots of the two terms that are perfect squares and write the factorization as the sum and difference of the square roots.

If subtraction separates two squared terms, then the product of the sum and difference of the two square roots factors the binomial: $a^2 - b^2 = (a + b)(a - b)$.

Factor $9x^2 - 16$.

The square roots of $9x^2$ and 16 are $3x$ and 4, respectively. The sum of the roots is $3x + 4$ and the difference between the roots is $3x - 4$. So, $9x^2 - 16 = (3x + 4)(3x - 4)$.

Factor $25z^2 - 81y^2$.

The square roots of $25z^2$ and $81y^2$ are $5z$ and $9y$, respectively. So, $25z^2 - 81y^2 = (5z + 9y)(5z - 9y)$.

Factor $x^4 - y^6$.

The square roots of x^4 and y^6 are x^2 and y^3, respectively. So the factorization of $x^4 - y^6 = (x^2 + y^3)(x^2 - y^3)$.

Creating factors for the difference of perfect cubes

A *perfect cube* is the number you get when you multiply a number times itself and then multiply the answer times the first number again. A cube is the third power of a number or variable. The difference of two cubes is a binomial expression $a^3 - b^3$.

To factor the difference of two perfect cubes, use the following pattern: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Here are the results of factoring the difference of perfect cubes:

- A binomial factor $(a - b)$ made up of the two cube roots of the perfect cubes separated by a minus sign.
- A trinomial factor $(a^2 + ab + b^2)$ made up of the squares of the two cube roots from the first factor added to the product of the cube roots in the middle. ***Remember:*** A trinomial has three terms, and this one has all plus signs in it.

Factor $64x^3 - 27y^6$.

The cube root of $64x^3$ is $4x$, and the cube root of $27y^6$ is $3y^2$. The square of $4x$ is $16x^2$, the square of $3y^2$ is $(3y^2)^2 = 9y^4$, and the product of $(4x)(3y^2)$ is $12xy^2$.

$$64x^3 - 27y^6 = (4x - 3y^2)(16x^2 + 12xy^2 + 9y^4)$$

Finishing with the sum of perfect cubes

You have a break coming. The rule for factoring the sum of two perfect cubes is almost the same as the rule for factoring the difference between perfect cubes, which I cover in the previous section. You just have to change two little signs to make it work.

To factor the sum of two perfect cubes, use the following pattern: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Factor $1{,}000z^3 + 343$.

The cube root of $1{,}000z^3$ is $10z$, and the cube root of 343 is 7. The product of $10z$ and 7 is $70z$. So, $1{,}000z^3 + 343 = (10z + 7)(100z^2 - 70z + 49)$.